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ON DOUBLING PROPERTIES FOR NON-NEGATIVE WEAK SOLUTIONS OF ELLIPTIC AND PARABOLIC PDE

BY

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ABSTRACT

In this paper we study quantitative properties of non-negative (super) solutions for elliptic and parabolic partial differential equations of second order with strongly singular coefficients, in which we cannot expect Harnack's inequality in general. We show the doubling property of u^δ with some small exponent $1 > \delta > 0$ for non-negative weak supersolutions u. Furthermore, we show the doubling property of u^q with large exponent $2(n+2)/n \geq q > 0$ for non-negative weak solutions u of parabolic equations.

1. Main results

In this paper we study quantitative properties for non-negative weak supersolutions of the following parabolic and elliptic partial differential equations:

$$\partial_t u - \mathrm{div}\mathcal{A}(t,x,u,\nabla u) + \mathcal{B}(t,x,u,\nabla u) = 0 \quad \text{in } Q = (0,T) \times \Omega, \tag{1}$$

$$-\mathrm{div}\mathcal{A}(x,u,\nabla u) + \mathcal{B}(x,u,\nabla u) = 0 \quad \text{in } \Omega, \tag{2}$$

where Ω is a domain in $\mathbb{R}^n$ with $n \geq 3$. For non-negative solutions it is well-known that the so-called Harnack's inequality holds if coefficients have weak singularities (see, e.g., [Ku1], [Za], [AS]). The purpose of this paper is to establish the doubling property of u^δ (Theorems 1.1 and 1.3) with some small exponent $1 > \delta > 0$ for non-negative weak supersolutions u instead of Harnack's inequality when coefficients have strong singularity. Furthermore, we establish the doubling

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property of u^q (Theorems 3.1 and 4.1) with larger exponents $q > 0$ for non-negative weak solutions u. These doubling property imply a kind of unique continuation property (Theorems 1.2 and 1.4).

In the parabolic case, we assume $\mathcal{A}(t,x,u,\xi)$ and $\mathcal{B}(t,x,u,\xi)$ are measurable with respect to t, x, continuous with respect to u, ξ, and satisfy the following structure conditions.

(A.1) There exist constants $0 < \lambda \le \Lambda < +\infty$ such that for every $(t,x,u,\xi) \in Q \times \mathbb{R} \times \mathbb{R}^n$
(i) $|\mathcal{A}(t,x,u,\xi)| \le \Lambda|\xi|, \quad \mathcal{A}(t,x,u,\xi)\cdot\xi \ge \lambda|\xi|^2$,
(ii) $|\mathcal{B}(t,x,u,\xi)| \le V(t,x)|u| + b(t,x)|\xi|$,
(iii) $V(t,x)$ and $b(t,x)$ are measurable and satisfy either (3) or (4):

$$\tag{3} \sup_{s\in(0,T)} \sup_{z\in\mathbb{R}^n, r>0} r^2\left(\frac{1}{|B_r(z)|}\int_{B_r(z)\cap\Omega} (V(s,x)+b^2(s,x))^t\,dx\right)^{1/t} \le K_t < +\infty, \quad 1 < \exists t \le n/2,$$

$$\tag{4} \sup_{s\in(0,T)} \sup_{z\in\mathbb{R}^n, r>0} \int_{B_r(z)\cap\Omega} \frac{(V(s,x)+b^2(s,x))}{|x-z|^{n-2}}\,dx \le S < +\infty.$$

In the elliptic case, we assume

(A.2) Let $1 < p < n$. There exist constants $0 < \lambda \le \Lambda < +\infty$ and weight $w(x) \ge 0$ such that for every $(x,u,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$
(i) $|\mathcal{A}(x,u,\xi)| \le \Lambda w(x)|\xi|^{p-1}, \quad \mathcal{A}(x,u,\xi)\cdot\xi \ge \lambda w(x)|\xi|^p$,
(ii) $|\mathcal{B}(x,u,\xi)| \le V(x)|u|^{p-1} + b(x)|\xi|^{p-1}$,
(iii) $w(x) \in A^p$-weight, $V(x)$ and $b(x)$ are measurable and satisfy either (5) or (6):

$$\tag{5} \sup_{z\in\mathbb{R}^n, r>0} r^p\left(\frac{1}{w(B_r(z))}\int_{B_r(z)\cap\Omega} \left(\frac{V(x)}{w(x)} + \left(\frac{b(x)}{w(x)}\right)^p\right)^t w(x)\,dx\right)^{1/t} \le K_t < +\infty, \quad 1 < \exists t \le n/p,$$

$$\tag{6} p = 2, \quad w(x) \equiv 1 \quad \text{and} \quad \sup_{z\in\mathbb{R}^n, r>0} \int_{B_r(z)\cap\Omega} \frac{(V(x)+b^2(x))}{|x-z|^{n-2}}\,dx \le S < +\infty.$$

For the definition of A_p-weight we refer to [GR].

Definition 1.1: (1) We say u is a non-negative weak supersolution of (1), if $u \geq 0, u \in L^2((0,T); W^{1,2}_{\text{loc}}(\Omega))$, $\partial_t u \in L^2((0,T); L^2(\Omega))$, and for every ϕ satisfying $\phi \geq 0, \phi \in L^2((0,T); W^{1,2}_o(\Omega))$, u satisfies

$$\int_0^T \int_\Omega \phi \partial_t u + \mathcal{A}(t,x,u,\nabla u) \cdot \nabla \phi + \mathcal{B}(t,x,u,\nabla u)\phi \, dxdt \geq 0. \tag{7}$$

(2) We say u is a non-negative weak supersolution of (2), if $u \geq 0, u \in W^{1,p}_{\text{loc}}(\Omega; w)$ and, for every ϕ satisfying $\phi \geq 0, \phi \in C^\infty_o(\Omega)$, u satisfies

$$\int_\Omega \mathcal{A}(x,u,\nabla u) \cdot \nabla \phi + \mathcal{B}(x,u,\nabla u)\phi \, dx \geq 0. \tag{8}$$

Notation: We use the following notation in the parabolic case:

$$\begin{aligned}
C_r &= \{(t,x) \in Q \,|\, |x - x_o| < r, t_o - 2r^2 < t < t_o\}, \\
D_r^+ &= \{(t,x) \in Q \,|\, |x - x_o| < r,\ t_o - r^2/2 < t < t_o\}, \\
D_r^- &= \{(t,x) \in Q \,|\, |x - x_o| < r,\ t_o - 2r^2 < t < t_o - 3r^2/2\},
\end{aligned}$$

where (t_o, x_o) is a fixed point in Q and $|x| = \max_{1 \leq i \leq n} |x_i|$; $|Q|, |D_r^\pm|$ are the $(n+1)$-dimensional Lebesgue measure. In the elliptic case we denote by $B_r = B_r(x_0)$ a ball with radius r and center x_0. We also use the notation

$$\fint_A f(x)\, dx = \frac{1}{|A|} \int_A f(x)\, dx.$$

Now we state the main results of this paper.

THEOREM 1.1: *Assume* (A.1). *Let u be a non-negative weak supersolution of (1). Then there exist constants $\delta \in (0,1)$ and $C > 0$ such that*

$$\iint_{D_r^-} u^\delta \, dxdt \leq C \iint_{D_{r/2}^+} u^\delta \, dxdt \tag{9}$$

for every $r > 0$ with $C_{2r} \subset Q$. The constants δ and C do not depend on r, u.

Theorem 1.1 implies the doubling property with small exponent $\delta \in (0,1)$ for non-negative weak supersolutions u. For non-negative weak solutions u, we will establish the doubling property with larger exponents q of u^q than Theorem 1.1 (see Theorem 3.1). As a corollary of Theorem 1.1 we obtain the following type of unique continuation theorem.

THEOREM 1.2: *Under the same assumption as in Theorem 1.1 we have the following: if for some* $(x_o, t_o) \in Q$ *and for every* $m > 0$

$$\iint_{D_r(x_o,t_o)} u\,dxdt = O(r^m) \quad \text{as } r \to 0 \tag{10}$$

holds, then $u(t,x) \equiv 0$ *for* $x \in \Omega, t \le t_o$.

Here we have used the notation

$$D_r(x_o,t_o) = \{(t,x) \in Q \,|\, |x - x_o| < r, t_o - 5r^2/2 < t < t_o - 2r^2\}.$$

Note that $D_r(x_o,t_o) \to (x_o,t_o)$ as $r \to 0$.

Similar results hold for elliptic equations (2).

THEOREM 1.3: *Assume* (A.2). *Let* u *be a non-negative weak supersolution of (2). Then there exist constants* $\delta \in (0,1)$ *and* $C > 0$ *such that*

$$\int_{B_r} u^\delta\,dx \le C \int_{B_{r/2}} u^\delta\,dx \tag{11}$$

for every $r > 0$ *with* $B_{2r} \subset \Omega$. *The constants* δ *and* C *do not depend on* r, u.

Remark 1.1: In Theorems 1.1 and 1.3, the constant δ is proportional to the quantity $1/(1+\Theta)^{1/p}, \Theta = K_1$ or S. In conditions (3) and (5), we assume $K_t < +\infty$ for some $t > 1$. However, in the case $w(x) \equiv 1$, if we assume an additional condition $V(x) + b^p(x) \in A_\infty$-weight (for the definition of A_∞ see [GR]), $K_1 < +\infty$ is sufficient (see e.g. [Sch], [Pe]).

THEOREM 1.4: *Under the same assumption as in Theorem 1.3 we have the following: if for some* $x_o \in \Omega$ *and for every* $m > 0$

$$\int_{B_r(x_o)} u\,dx = O(r^m) \quad \text{as } r \to 0 \tag{12}$$

holds, then $u(x) \equiv 0$ *for* $x \in \Omega$.

It is well-known that Theorem 1.3 implies Theorem 1.4 (e.g. [CG]). Theorems 1.3 and 1.4 were first proved by Chiarenza and Garofalo in [CG] for the special case $-\operatorname{div}(A(x)\nabla u) + V(x)u = 0$, $A(x)$ is uniformly elliptic and V belongs to the Lorentz space $L^{n/2,\infty}$. Theorems 1.3 and 1.4 are the generalization of [CG] to the degenerate case under general structure condition (A.2). We also note that the condition (5) in (A.2) includes Lorentz space $L^{n/2,\infty}$ in the non-degenerate case. As far as I know, the estimates for parabolic equations in Theorem 1.1 and Theorem 3.1 and the estimate in Theorem 4.1 for elliptic equations are new even for linear equations.

Remark 1.2: Theorems 1.1 and 1.3 are focused on the case that coefficients have strong singularity, for example $V(x) = 1/|x|^2, |\mathbf{b}(x)| = 1/|x|$ for the linear equations $-\mathrm{div}(A(x)\nabla u) + \mathbf{b}\cdot\nabla u + Vu = 0$. Under assumptions (A.1) and (A.2) Harnack's inequality does not hold in general and, moreover, solutions might be unbounded. Note that if V, b have weaker singularities than (A.1) or (A.2), Harnack's inequality holds (e.g. [Ku1], [Za]) and doubling estimates (9) and (11) are an easy consequence of it.

Remark 1.3: We cannot expect the estimates in the Theorems above under general assumptions (A.1) or (A.2), if we remove the non-negativity of solutions $u \geq 0$. For example, without the condition $u \geq 0$ it is known that to obtain unique continuation theorems for $L = -\mathrm{div}(A(x)\nabla u) + \mathbf{b}\cdot\nabla u + Vu = 0$, $A(x)$ must be Lipschitz continuous for $n \geq 3$. Moreover, Wolff recently showed that the strong unique continuation theorem does not hold in general in the case $|\mathbf{b}| \in L^n$. Various authors (e.g., N. Aronszajn, D. Jerison–C. Kenig, C. D. Sogge, T. Wolff, N. Garofalo–F. H. Lin, etc.) tried to obtain the unique continuation theorem under minimal assumptions on coefficients. Recent progress and references on this matter in the elliptic case can be seen in [Wo]. We also note that similar doubling estimates as in Theorems 1.1 and 1.3 can be seen in [GL1,2] and [Ku2,3] without non-negativity of solutions.

In section 2 we prove Theorems 1.1 and 1.2. In section 3 we show furthermore the doubling property of $u^q\,dxdt$ with $0 < q \leq 2(n+2)/n$ for non-negative weak solutions u of parabolic equations. In section 4 we mention the proof of Theorem 1.3 briefly and show the doubling property of $u^q\,dx$ with $0 < q \leq 2n/(n-2)$ for non-negative weak solutions u of elliptic equations.

2. Proof of Theorems 1.1 and 1.2

In this section we give the proof of Theorems 1.1 and 1.2.

LEMMA 2.1: *Assume $\sigma = \sigma(x) \in A_2$ and that there exists a constant C such that*

$$\left(\frac{1}{\sigma(D_r^-)}\int\int_{D_r^-} v\sigma\,dxdt\right)\left(\frac{1}{\sigma(D_r^+)}\int\int_{D_r^+} v^{-1}\sigma\,dxdt\right) \leq C$$

for every $r > 0$ and D_r^-, D_r^+ with $C_{2r} \subset Q$. Then we have

$$\left(\frac{1}{\sigma(D_r^-)}\int\int_{D_r^-} v\sigma\,dxdt\right) \leq C\left(\frac{1}{\sigma(D_{r/2}^+)}\int\int_{D_{r/2}^+} v\sigma\,dxdt\right). \tag{13}$$

Proof: $\sigma \in A_2$ implies $\sigma(B_r) \leq C\sigma(B_{r/2})$. Hence $\sigma(D_r^+) \leq C\sigma(D_{r/2}^+)$. By the assumption we have

$$\frac{1}{\sigma(D_r^+)}\int\int_{D_r^+} f\sqrt{v}\frac{1}{\sqrt{v}}\sigma\,dxdt \le \left(\frac{1}{\sigma(D_r^+)}\int\int_{D_r^+} f^2 v\sigma\,dxdt\right)^{1/2} \cdot\left(\frac{1}{\sigma(D_r^+)}\int\int_{D_r^+} v^{-1}\sigma\,dxdt\right)^{1/2}$$
$$\le C\left(\frac{1}{\sigma(D_r^+)}\int\int_{D_r^+} f^2 v\sigma\,dxdt\right)^{1/2} \cdot\left(\frac{\sigma(D_r^-)}{\int\int_{D_r^-} v\sigma\,dxdt}\right)^{1/2}.$$

Taking $f = \chi_{D^+_{r/2}}$,

$$\text{LHS} = \frac{\sigma(D^+_{r/2})}{\sigma(D_r^+)} \quad \text{and}$$
$$\text{RHS} = \left(\frac{1}{\sigma(D_r^+)}\int\int_{D^+_{r/2}} v\sigma\,dxdt\right)^{1/2}\left(\frac{\sigma(D_r^-)}{\int\int_{D_r^-} v\sigma\,dxdt}\right)^{1/2}.$$

Thus we obtain

$$\sqrt{\frac{\sigma(D^+_{r/2})}{\sigma(D_r^+)}}\left(\frac{1}{\sigma(D_r^-)}\int\int_{D_r^-} v\sigma\,dxdt\right)^{1/2} \le C\left(\frac{1}{\sigma(D^+_{r/2})}\int\int_{D^+_{r/2}} v\sigma\,dxdt\right)^{1/2}$$

and we have the desired estimate. ∎

Let $U = \{(t,x) |\ |t| < 1,\ |x| < 1\}$ and let $C^+ = \{|x-a| < \rho,\ 0 < t - t_0 < \rho^2\}$ and $C^- = \{|x-a| < \rho,\ 0 < t_0 - t < \rho^2\}$ be subdomains in U for a fixed $(t_0, a) \in U$. Define $\Psi(s) = \sqrt{s}$ if $s > 0$ and 0 if $s \le 0$. The following lemma plays an important role in the proof of Theorem 1.1.

LEMMA 2.2: *Let $\epsilon > 0$ and u be a non-negative weak supersolution of (1) on $U' = \{|t| < 1,\ |x| < 2\}$. Put $v = -\log(u+\epsilon)$. Then we have*

$$\frac{1}{|C^+||C^-|}\int\int_{x'\in C^+, y'\in C^-} \Psi(v(x') - v(y'))\,dx'dy' \le C. \tag{14}$$

Here C is a constant independent of $(t_0, a) \in U$ and ρ.

The proof of Lemma 2.2 relies on the argument of Moser [M]. We first recall the following lemma.

LEMMA 2.3 (M, Lemma 3, p. 120): *Let $p(x)$ be a positive continuous function with* supp $p \subset R_r = \{|x| \le r\}$. *Suppose the level sets $\{x \in R_r; p(x) \ge c\}$ are convex for each positive constant c. Then we have*

$$\int_{R_r} (f-\overline{f})^2 p\,dx \le Cr^2 \int_{R_r} |\nabla f|^2 p\,dx \tag{15}$$

for every $f \in H^1(R_r)$, where $\overline{f} = (\int fp\,dx)/(\int p\,dx)$.

Proof of Lemma 2.2: Let $\phi(x) = \Pi_{i=1}^n \chi_i(x_i)$, $\chi_i(x_i) = 1$ on $|x_i| < 1$, 0 on $|x_i| > 2$, and χ_i is linear on $1 < |x_i| < 2$. Then the level sets of ϕ^2 are convex. Let $-1 < t_1 < t_2 < 1$ and $\eta_n(t)$ be a cut-off function satisfying $\eta_n(t) = 1$ on $[t_1, t_2]$, 0 on $(t_2 + 1/n, 1) \cup (-1, t_1 - 1/n)$. Taking $\Phi = \phi(x)^2\eta_n(t)(u+\epsilon)^{-1}$ as a test function in (7) and letting $n \to \infty$, we obtain

$$\int_{R_2} \phi^2 v\,dx\Big|_{t=t_1}^{t=t_2} + \lambda \int_{t_1}^{t_2}\int_{R_2} \frac{|\nabla u|^2\phi^2}{(u+\epsilon)^2}\,dxdt \le$$
$$2\Lambda \int_{t_1}^{t_2}\int_{R_2} \frac{|\nabla u||\phi||\nabla\phi|}{(u+\epsilon)}\,dxdt + \int_{t_1}^{t_2}\int_{R_2} \frac{V|u||\phi|^2}{(u+\epsilon)}\,dxdt + \int_{t_1}^{t_2}\int_{R_2} \frac{b|\nabla u||\phi|^2}{(u+\epsilon)}\,dxdt.$$

By Schwarz's inequality there exists a constant $C > 0$ such that

$$\begin{aligned}\int_{R_2} \phi^2 v\,dx\Big|_{t=t_1}^{t=t_2} &+ \frac{\lambda}{2}\int_{t_1}^{t_2}\int_{R_2} |\nabla v|^2\phi^2\,dxdt \\ &\le C\int_{t_1}^{t_2}\int_{R_2} (1+\phi^2(b^2+V))dxdt \le C(1+K_1)(t_2-t_1)|R_2|.\end{aligned}$$

Applying Lemma 2.3 to $p = \phi^2$ and $\overline{V}(t) = (\int_{R_2} v(t,x)\phi^2(x)\,dx)/(\int_{R_2}\phi^2(x)\,dx)$, it follows that

$$(\overline{V}(t_2)-\overline{V}(t_1))\int_{R_2}\phi^2\,dx + \frac{\lambda}{2C}\int_{t_1}^{t_2}\int_{R_2}(v-\overline{V}(t))^2\phi^2\,dxdt \le C(1+K_1)(t_2-t_1)|R_2|.$$

Since $|R_2|/\int_{R_2}\phi^2\,dx$ is comparable to a positive constant,

$$\frac{\overline{V}(t_2)-\overline{V}(t_1)}{t_2-t_1} + \frac{C}{|R_2|(t_2-t_1)}\int_{t_1}^{t_2}\int_{R_2}(v(t,x)-\overline{V}(t))^2\phi^2(x)\,dxdt \le C.$$

Since $\overline{V}(t)$ is absolutely continuous, taking $t_2 \to t_1$, we obtain

$$\frac{d\overline{V}(t)}{dt} + \frac{C}{|R_1|}\int_{R_1}(v(t,x)-\overline{V}(t))^2\,dx \le C, \quad |t| < 1. \tag{16}$$

We may assume $\overline{V}(0) = 0$, because the estimate (16) does not change if we replace v by $v+$ constant. Once we obtain this estimate we can conclude the desired estimate by the argument as in [M, pp. 122–123]. ∎

Proof of Theorem 1.1: Let $D^+ = \{|x-a| < \rho,\ \rho^2/2 < t - t_0 < \rho^2\}$ and $D^- = \{|x-a| < \rho,\ \rho^2/2 < t_0 - t < \rho^2\}$. Once we obtain the estimate in Lemma 2.2, we have the following John-Nirenberg type estimate by the main lemma in [M, p. 106]:

$$\frac{1}{|D^+||D^-|}\int\int_{x'\in D^+, y'\in D^-} \Phi(v(x') - v(y'))\,dx'dy' \le 1, \tag{17}$$

where $\Phi(s) = Ce^{\alpha s}$ for some α and C depending only on n. By the transformations $t \to a^2 t + t_0$ and $x_k \to a x_k + x_{o,k}$, we can conclude that (17) holds for every D_r^+, D_r^- with $C_{2r} \subset Q$. Therefore there exists a constant $\delta > 0$ such that

$$\left(\frac{1}{r^{n+2}}\int\int_{D_r^-}(u+\epsilon)^{\delta}\,dxdt\right)\left(\frac{1}{r^{n+2}}\int\int_{D_r^+}(u+\epsilon)^{-\delta}\,dxdt\right) \le C$$

for every $r > 0$ and D_r^-, D_r^+ with $C_{2r} \subset Q$. Taking $\epsilon \to 0$ we have

$$\left(\frac{1}{|D_r^-|}\int\int_{D_r^-} u^{\delta}\,dxdt\right)\left(\frac{1}{|D_r^+|}\int\int_{D_r^+} u^{-\delta}\,dxdt\right) \le C. \tag{18}$$

By Lemma 2.1 with $\sigma(x) \equiv 1$ we can conclude

$$\int\int_{D_r^-} u^{\delta}\,dxdt \le C\int\int_{D_{r/2}^+} u^{\delta}\,dxdt. \qquad ∎$$

Proof of Theorem 1.2: By Theorem 1.1 we have

$$\int\int_{D_r(t_o,x_o)} u^{\delta}\,dxdt \le C^l \int\int_{D_{\frac{r}{2^l}}(t_o,x_o)} u^{\delta}\,dxdt$$

$$\le r^{m+n+2}\left(\frac{C}{2^{m+n+2}}\right)^l \left(\frac{r}{2^l}\right)^{-m}\left(\frac{r}{2^l}\right)^{-n-2}\int\int_{D_{\frac{r}{2^l}}(t_o,x_o)} u^{\delta}\,dxdt \tag{19}$$

for each $l, m \in \mathcal{N}$. Choose $m > 0$ such that $C \le 2^{m+n+2}$. Then

$$\int\int_{D_r(t_o,x_o)} u^{\delta}\,dxdt \le r^{m+n+2}\left\{\left(\frac{r}{2^l}\right)^{-\frac{m}{\delta}-n-2}\int\int_{D_{\frac{r}{2^l}}(t_o,x_o)} u\,dxdt\right\}^{\delta}$$

$$\to 0 \quad (l \to \infty).$$

Hence we obtain $u(t,x) \equiv 0$ on $D_r(t_o,x_o)$. Since $r > 0$ is arbitrary, u must vanish on a backward parabolic region with respect to the point (t_o, x_o). By using the argument above for different points (t_o', x_o'), we can attain the desired result. ∎

3. Doubling property for parabolic equations

In this section we study the doubling property of $u^q\,dxdt$ for large exponent q in the case that $u \geq 0$ is not only a weak supersolution of (1) with (A.1) but also a subsolution of

$$\partial_t u - div\mathcal{A}(t,x,u,\nabla u) + \mathcal{B}'(t,x,u,\nabla u) \leq 0 \tag{20}$$

for $\mathcal{B}'$ satisfying

$$|\mathcal{B}'(t,x,u,\xi)| \leq W(t,x)|u| + c(t,x)|\xi|.$$

We assume the following for $W + c^2$:

(A.3) We can write $W(t,x) + c^2(t,x) = Q_1(t,x) + Q_2(t,x), Q_j(t,x) \geq 0$ $(j = 1,2)$ and $Q_j(x)$ satisfy
(21)

$$\sup_{s\in(0,T)} \sup_{z\in\Omega, r>0} r^2 \left(\frac{1}{|B_r(z)|} \int_{B_r(z)\cap\Omega} Q_1(s,x)^t\,dx \right)^{1/t} \leq \epsilon_0 \quad \text{for some } 1 < t \leq n/2,$$

$$\sup_{s\in(0,T)} \limsup_{r\to 0} \sup_{z\in\Omega} \int_{B_r(z)\cap\Omega} \frac{Q_2(s,x)}{|x-z|^{n-2}}\,dx \leq \epsilon_0. \tag{22}$$

Here ϵ_0 is a sufficiently small constant depending only on n, λ, Λ and Ω.

THEOREM 3.1: *Let u be a non-negative weak supersolution of (1) with* (A.1) *and a weak subsolution of (20) with* (A.3) *and let $0 < q \leq 2(n+2)/n$. Then there exist constants $0 < l' < l, 0 < m$, $0 < m'' < m'$, $L > 0$ and C such that*

$$\int\int_{Q_r^-} u^q\,dxdt \leq C \int\int_{Q_r^+} u^q\,dxdt \tag{23}$$

for every $r > 0$ with $Q_{Lr}^+ \subset Q$. Here

$$Q_r^- = \{|x - x_o| < lr, t_o - (m+m')r^2 < t < t_o - m'r^2\},$$
$$Q_r^+ = \{|x - x_o| < l'r, t_o - m''r^2 < t < t_o\}.$$

COROLLARY 3.1: *Let $0 < q \leq 2(n+2)/n$ and let u be a non-negative weak solution of*

$$\partial_t u - \operatorname{div}(A(t,x)\nabla u) + \mathbf{b}(t,x)\cdot\nabla u + V(t,x)u = 0$$

in $Q = (0,T)\times\Omega$, where Ω is a bounded domain in $\mathbb{R}^n, n \geq 3$. Assume $A(t,x)$ is uniformly elliptic, V^+ satisfies (3) or (4) and $V^- + |\mathbf{b}|^2$ satisfies (A.3). *Then the same estimate as in Theorem 3.1 holds.*

First, we note the following Sobolev-type inequality.

LEMMA 3.1: *Let $q = 2(n+2)/n$ and $Q_r = \{|x - x_0| < r, t_0 - r^2/2 < t < t_0\}$. Then we have*

$$(24)\quad \int\int_{Q_r} |u(t,x)|^q \, dxdt \leq C\left(r^2 \int\int_{Q_r} |\nabla u|^2 \, dxdt + \int\int_{Q_r} |u|^2 \, dxdt\right) \cdot \left(r^2 \sup_{t \in (t_0 - r^2/2, t_0)} ||u(t,\cdot)||_{L^2(B_r(x_0))}\right)^{4/n}.$$

Proof: This inequality essentially can be seen in [M; Lemma 2]. We may assume $r = 1$. By Sobolev's inequality

$$||u(t,\cdot)||_q \leq C(||\nabla u(t,\cdot)||_2^2 + ||u(t,\cdot)||_2^2)^{n/2(n+2)} ||u(t,\cdot)||_2^{2/(n+2)}$$

holds for every t. Hence we arrive at the desired estimate. ∎

We also note the following Caccioppoli-type inequality.

LEMMA 3.2: *Under the same assumptions as in Theorem 3.1, there exist a constant C depending only on n, λ, Λ such that*

$$(25)\quad \int\int_{R'} |\nabla u|^2 \, dxdt \leq C\left(\frac{1}{(\rho - \rho')^2} + \frac{1}{(\tau - \tau')}\right) \int\int_R |u|^2 \, dxdt,$$

$$(26)\quad \max_{t \in (-\tau', 0)} \int |u|^2(t,x) \, dx \leq C\left(\frac{1}{(\rho - \rho')^2} + \frac{1}{(\tau - \tau')}\right) \int\int_R |u|^2 \, dxdt,$$

where

$$R = \{|x| < \rho, \ -\tau < t < 0\}, \quad R' = \{|x| < \rho', \ -\tau' < t < 0\},$$

$0 < \rho' < \rho, \ 0 < \tau < \tau'$.

Proof: This was shown in the case $b = V \equiv 0$ in [M, Eq. (3.3) and (3.4)]), but the same proof works under assumptions (A.1) and (A.3), since we can control lower order terms by using well-known weighted norm inequalities (see, e.g., Lemma 4.1 below and [Ku1,3], [Pe], [Sch]). We omit the details. One can see the detailed computations in [AS, Eq. (35)] for equations with somewhat mild singular coefficients. ∎

Furthermore, we establish the following reverse Hölder inequality by using Lemma 3.1 and the argument in [CFG].

LEMMA 3.3: *Let $0 < p < 2$ and $0 < \tau < 1$. Then there exists a constant $C = C(n, \lambda, \Lambda, p, \tau)$ such that*

$$\left(\frac{1}{|Q_{\tau r}|}\int\int_{Q_{\tau r}} |u|^2\,dxdt\right)^{1/2} \le C\left(\frac{1}{|Q_r|}\int\int_{Q_r} |u|^p\,dxdt\right)^{1/p} \tag{27}$$

for every $Q_{2r} \subset Q$.

Proof: By a scaling argument, it suffices to show the case $r = 1$. Let $0 < p < 2$ and $I(s) = (\int\int_{Q_s} |u|^2\,dxdt)^{1/2}$ with $\tau \le s \le 1$. We shall show that $I(\tau)$ is bounded under the assumption $\int\int_{Q_1} |u|^p\,dxdt = 1$. We may assume $I(\tau) > 1$, otherwise there is nothing to be done. Hence $I(s) > 1$ for $s \ge \tau$. Choose $\theta \in (0,1)$ such that $(2-\theta p)/(1-\theta) = 2(n+2)/n$. Using Hölder's inequality for $I(s) = (\int\int_{Q_s} |u|^{2-\theta p}|u|^{\theta p}\,dxdt)^{1/2}$, we have

$$\begin{aligned} I(s) &\le \left(\int\int_{Q_s} |u|^{\frac{2-\theta p}{1-\theta}}\,dxdt\right)^{\frac{1-\theta}{2}}\left(\int\int_{Q_s} |u|^p\,dxdt\right)^{\frac{\theta}{2}} \\ &\le \left(\int\int_{Q_s} |u|^{\frac{2(n+2)}{n}}\,dxdt\right)^{\frac{1-\theta}{2}}. \end{aligned}$$

Lemma 3.1 and Lemma 3.2 yield

$$\begin{aligned} I(s) &\le C\left(\frac{1}{(t-s)^2}\int\int_{Q_t} |u|^2\,dxdt\right)^{\frac{1-\theta}{2}}\left(\frac{1}{(t-s)^2}\int\int_{Q_t} |u|^2\,dxdt\right)^{\frac{2}{n}\frac{1-\theta}{2}} \\ &= C\left(\frac{I(t)}{t-s}\right)^{\theta^*} \end{aligned} \tag{28}$$

for $\tau \le s < t \le 1$, where $\theta^* = (1+\frac{2}{n})(1-\theta) < 1$. Set $s = t^b$ for some $b > 1$ with $\tau \le t^b$. Then it follows that

$$\frac{\log I(t^b)}{t} \le \frac{\log C}{t} + \theta^* \frac{1}{t}\log\frac{1}{(t-t^b)} + \theta^* \frac{\log I(t)}{t}.$$

Integrating over $(\tau^{1/b}, 1)$, we have

$$\int_{\tau^{1/b}}^1 \frac{\log I(t^b)}{t}\,dt \le C' + \theta^* \int_{\tau^{1/b}}^1 \frac{\log I(t)}{t}\,dt.$$

Since $\tau < \tau^{1/b}$ and $\log I(t) \ge 0$, it follows that

$$\int_{\tau^{1/b}}^1 \frac{\log I(t)}{t}\,dt \le \int_{\tau}^1 \frac{\log I(t)}{t}\,dt.$$

By the change of variable $u = t^b$,

$$\int_{\tau^{1/b}}^{1} \frac{\log(t^b)}{t}\,dt = \frac{1}{b}\int_{\tau}^{1} \frac{\log(u)}{u}\,du$$

holds. Hence we obtain

$$\left(\frac{1}{b} - \theta^*\right)\int_{\tau}^{1} \frac{\log I(t)}{t}\,dt \le C. \tag{29}$$

Taking $b > 1$ such that $\frac{1}{b} - \theta^* > 0$, we can conclude $\log I(\tau) \le C$. ∎

By combining Theorem 1.1, the Caccioppoli inequality and Lemma 3.1 we can complete the proof of Theorem 3.1.

Proof of Theorem 3.1: Let $\delta > 0$ be the constant in Theorem 1.1. For $\delta < q \le 2$, Lemma 3.3 with $\tau = \frac{2}{3}$ and Theorem 1.1 yield

$$\begin{aligned}\left(\frac{1}{r^{n+2}}\int\!\!\int_{D_r^1} u^q\,dxdt\right)^{1/q} &\le C\left(\frac{1}{r^{n+2}}\int\!\!\int_{D_r^2} u^\delta\,dxdt\right)^{1/\delta}\\ &\le C\left(\frac{1}{r^{n+2}}\int\!\!\int_{D_r^3} u^\delta\,dxdt\right)^{1/\delta},\end{aligned}$$

where

$$\begin{aligned}D_r^1 &= \{|x - x_o| < 2r/3\sqrt{2},\ t_o - (\frac{3}{4} + \frac{1}{9})r^2 < t < t_o - \frac{3}{4}r^2\},\\ D_r^2 &= \{|x - x_o| < r/\sqrt{2},\ t_o - r^2 < t < t_o - \frac{3}{4}r^2\},\\ D_r^3 &= \{|x - x_o| < r/2\sqrt{2},\ t_o - r^2/16 < t < t_o\}.\end{aligned}$$

Hence by Hölder's inequality we obtain the doubling estimate of $u^q\,dxdt$ for $0 < q \le 2$:

$$\left(\frac{1}{r^{n+2}}\int\!\!\int_{D_r^1} u^q\,dxdt\right)^{1/q} \le C\left(\frac{1}{r^{n+2}}\int\!\!\int_{D_r^3} u^q\,dxdt\right)^{1/q}. \tag{30}$$

On the other hand, Lemma 3.1 and the Caccioppoli-type inequality yield the following reverse Hölder-type estimate:

$$\begin{aligned}&\left(\frac{1}{r^{n+2}}\int\!\!\int_{Q_r} u^{\frac{2(n+2)}{n}}\,dxdt\right)^{\frac{n}{2(n+2)}}\\ \le\, &C\left(\frac{1}{r^n}\int\!\!\int_{Q_r} |\nabla u|^2\,dxdt + \frac{1}{r^{n+2}}\int\!\!\int_{Q_r} |u|^2\,dxdt\right)^{\frac{n}{2(n+2)}}\\ &\cdot\left(\frac{1}{r^n}\sup_t \int_{B_r} u^2\,dxdt\right)^{\frac{1}{(n+2)}}\\ \le\, &C\left(\frac{1}{r^{n+2}}\int\!\!\int_{Q_{\frac{r}{\tau}}} u^2\,dxdt\right)^{1/2}\end{aligned} \tag{31}$$

for every $0 < \tau < 1$. Hence we obtain the doubling estimate for every $q \leq 2(n+2)/n$ by using Hölder's inequality and the doubling estimate of $u^2\,dxdt$. ∎

4. Doubling property for elliptic equations

By using the following lemmas, we can prove Theorem 1.3 in a similar way as in the proof of Theorem 1.1. So we omit the details of the proof of Theorem 1.3.

LEMMA 4.1 (CW, Theorem 1.2): *Let $w \in A_p$ and $a(x) \geq 0$ satisfy*

$$\sup_{r>0,\ z\in\mathbb{R}^n} r^p\left(\frac{1}{w(B_r(z))}\int_{B_r(z)} a(x)^t w(x)\,dx\right)^{1/t} \leq K < \infty$$

for some $n/p \geq t > 1$, where $w(B_r(z)) = \int_{B_r(z)} w(x)\,dx$. Then

$$\int_{B_r(z)} a(x)|u|^p\,dx \leq C_0 K \int_{B_r(z)} |\nabla u|^p w(x)\,dx \tag{32}$$

holds for every $u \in C_o^\infty(B_r(z))$ and $r > 0$.

LEMMA 4.2 ([FKS]): *Let w be A_p-weight. Then we have*

$$\frac{1}{|B_r|}\int_{B_r} |u-(u)_r|\,dx \leq Cr\left(\frac{1}{w(B_r)}\int_{B_r} |\nabla u|^p w(x)\,dx\right)^{1/p}, \tag{33}$$
$$(u)_r = \frac{1}{|B_r|}\int_{B_r} u\,dx.$$

In this section, we study furthermore the doubling property of $u^q\,dx$ for large exponent q in the case that $u \geq 0$ is not only a supersolution of (2) satisfying (A.2) with $p = 2$ and $w(x) \equiv 1$, but also a subsolution of

$$-\mathrm{div}\mathcal{A}(x,u,\nabla u) + \mathcal{B}'(x,u,\nabla u) \leq 0 \quad \text{in } \Omega \tag{34}$$

for another $\mathcal{B}'$, where Ω is a bounded domain. We assume for $\mathcal{B}'(x,u,\xi)$ that

$$|\mathcal{B}'(x,u,\xi)| \leq W(x)|u| + c(x)|\xi|$$

and:

(A.4) We can write $W(x) + c^2(x) = Q_1(x) + Q_2(x)$, $Q_j(x) \geq 0$ $(j = 1,2)$ and $Q_j(x)$ satisfy

$$\sup_{z\in\Omega, r>0} r^2\left(\frac{1}{|B_r(z)|}\int_{B_r(z)\cap\Omega} Q_1(x)^t\,dx\right)^{1/t} \leq \epsilon_0 \quad \text{for some } 1 < t \leq n/2, \tag{35}$$

$$\limsup_{r\to 0}\sup_{z\in\Omega}\int_{B_r(z)\cap\Omega}\frac{Q_2(x)}{|x-z|^{n-2}}\,dx \leq \epsilon_0. \tag{36}$$

Here ϵ_0 is the same constant in (A.3) depending only on n, λ, Λ and Ω.

LEMMA 4.3: *Let u be a subsolution of (34). Assume* (A.4). *Then we have the Caccioppoli-type inequality:*

$$\int_{B_s}|\nabla u|^2\,dx \leq C\frac{1}{(t-s)^2}\int_{B_t}|u|^2\,dx \tag{37}$$

for every $0 < s < t$ with $B_t \subset \Omega$. Here $C = C(n, \lambda, \Lambda, \Omega)$.

Proof: This can be proved by using Lemma 4.1 with $w(x) \equiv 1$ as in [Ku1, Lemma 2.2]. ∎

PROPOSITION 4.1: *We assume that a non-negative function u satisfies the Caccioppoli-type inequality and the doubling condition with exponent $\delta \in (0,1)$. Then for every $0 < q \leq 2n/(n-2), r > 0$ with $B_{4r} \subset \Omega$, we have*

$$\int_{B_r} u^q\,dx \leq C\int_{B_{r/2}} u^q\,dx. \tag{38}$$

To prove Proposition 4.1 we note the following reverse Hölder-type inequality which can be proved in the same way as in Lemma 3.3.

LEMMA 4.4: *Let u satisfy the Caccioppoli-type inequality (37) and $0 < p < 2$. Then*

$$\left(\fint_{B_{r/2}}|u|^2\,dx\right)^{1/2} \leq C_1\left(\fint_{B_r}|u|^p\,dx\right)^{1/p} \tag{39}$$

holds for every $B_{2r} \subset \Omega$.

Proof of Proposition 4.1: First of all, the doubling property of $u^\delta\,dx$ and Lemma 4.4 imply that $u^2\,dx$ satisfies the doubling estimate:

$$\begin{aligned}\left(\fint_{B_{r/2}} u^2\,dx\right)^{1/2} &\leq C\left(\fint_{B_r} u^\delta\,dx\right)^{1/\delta} \\ &\leq C\left(C_o^2\fint_{B_{r/4}} u^\delta\,dx\right)^{1/\delta} \leq CC_o^{2/\delta}\left(\fint_{B_{r/4}} u^2\,dx\right)^{1/2}.\end{aligned}$$

The doubling property of $u^2\,dx$ and Lemma 4.4 assure that $u^q\,dx$ satisfies the doubling estimate for $0 < q < 2$. On the other hand, the Caccioppoli-type inequality and the doubling property of $u^2\,dx$ yield

$$\int_{B_r} |\nabla u|^2\,dx \le \frac{C}{r^2}\int_{B_{2r}} u^2\,dx \le \frac{C'}{r^2}\int_{B_r} u^2\,dx.$$

By Sobolev's inequality, we obtain

$$\left(\fint_{B_r} u^{2^*}\,dx\right)^{1/2^*} \le C\left(\fint_{B_r} u^2\,dx\right)^{1/2}, \quad 2^* = \frac{2n}{n-2}.$$

That is, u^2 satisfies the reverse Hölder inequality, and hence $u^2 \in A_\infty$. It is well-known that this implies $u^2 \in A_r$ for some $r > 1$ (see e.g. [GR, pp. 403–404]). Also now we conclude that $u^{2^*}\,dx$ satisfies the doubling property and hence $u^q\,dx$ satisfies the doubling property even for every $2 \le q \le 2^*$. ∎

Proposition 4.1 and Theorem 1.3 yield

THEOREM 4.1: *Let u be a non-negative weak solution of*

$$-\mathrm{div}(A(x)\nabla u) + \mathbf{b}(x)\cdot\nabla u + V(x)u = 0$$

in a bounded domain $\Omega \subset \mathbb{R}^n, n \ge 3$. Assume $A(x)$ is uniformly elliptic, V^+ satisfies (5) or (6) with $p = 2$ and $w(x) \equiv 1$, and $V^- + |\mathbf{b}|^2$ satisfies (A.4), *where $V = V^+ - V^-$, $V^+ = \max(V,0), V^- = \max(-V,0)$. Then for every $0 < q \le 2n/(n-2), r > 0$ with $B_{4r} \subset \Omega$, we have*

$$\int_{B_r} u^q\,dx \le C\int_{B_{r/2}} u^q\,dx. \tag{40}$$

Proof: Since $V^+ + |\mathbf{b}|^2$ satisfies (5) or (6) in (A.2) with $p = 2, w \equiv 1$, u is a supersolution of (2) with (A.2). Hence by Theorem 1.3 there exists a constant $\delta \in (0,1)$ such that $u^\delta\,dx$ satisfies the doubling estimate. On the other hand, u is a subsolution of (34) with (A.4). Therefore, Theorem 4.1 yields the desired estimate. ∎

5. Concluding remarks

1. Although in Theorem 1.1 we only deal with the non-degenerate case, we can obtain similar results even in the degenerate case for (1) under the assumption $w \in A_{1+2/n}$ with some modification of the set $D_r^\pm$. Here we assume the condition (A.2) (i) with $p = 2$ for $\mathcal{A}(t,x,u,\xi)$ and the condition (A.2) (ii), (iii) with $p = 2$

for $\mathcal{B}(t,x,u,\xi)$ (see [CS1]). We note that $A_{1+2/n} \subset A_2$, since it is well-known that $A_p \subset A_q$ for $p < q$. Moreover, for the equation

$$w(x)\partial_t u - \operatorname{div}(w(x)\nabla u) + \mathcal{B}(t,x,u,\nabla u) = 0 \tag{41}$$

we can obtain the same results on $D_r^{\pm}$ under the assumption $w \in A_2$ and the same assumptions for $\mathcal{B}(t,x,u,\xi)$ (see [CS2]).

2. We can replace the condition (3) in (A.1), which only allows a uniform singularity in the time variable, to the $L^{p,q}$-type condition, that is $|V| + |\mathbf{b}|^2 \in L^{p,q}(Q)$ with $n/2p + 1/q$ for some $p, q > 1$. We also mention that in the definition of the weak solution of (1) we assumed a somewhat strong condition $u_t \in L^2((0,T); L^2(\Omega))$. However, all the results of this paper would hold for general weak solutions $u \in L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); W^{1,2}(\Omega))$ by using the regularization argument in [AS, pp. 119–121].

3. Finally, we mention an application of Theorem 1.3 to a nonlinear equation. Let $u \in H^1 \cap L^p$ be a weak solution of $-\Delta u = u^p - u^q$, $u \geq 0$, $1 < q < p$, $n/(n-2) \leq p$. In general we do not know the regularity of u, for example in the supercritical case $p > (n+2)/(n-2)$. In [Pa] it was proved that the weak solution is regular if

$$\sup_{r>0,x} r^{-\lambda} \int_{B_r(x)} u^p \, dx \leq C \tag{42}$$

for $\lambda > n - 2p/(p-1)$. If we apply Theorem 1.3 directly, the constant $\delta > 0$ in the estimate depends on a certain norm of $V = -u^{p-1} + u^{q-1}$ or $V = u^{q-1}$, hence it might depend on a solution u. Actually, by using the method of [Pa], we can show that

$$r^{2p/(p-1)-n} \int_{B_r(x)} u^p \, dx \leq C \tag{43}$$

for every $1 \geq r > 0$ with $B_r(x) \subset \Omega$. This implies that $V = u^{p-1}$ and $V = u^{q-1}$ satisfy the condition (5) in (A.2) with $t = p/(p-1) \in (1, n/2]$ and K_t independent of u. Therefore, we obtain uniform estimate (11) for u; that is, the constant δ is independent of u.

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